9/10 Recitation

1. If the following statement is true, prove it; if it is false, provide a counterexample. If A and B are invertible $n \times n$ matrices, then so is $A + B$.

2. Use the Gauss-Jordan algorithm to find the inverse of $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$. Verify your answer with Julia.

3. Find a 2×2 matrix that its its own inverse (other than the identity). You might experiment using Julia or figure it out by hand.

4. Let $S, T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformations given by

$$S\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y \\ y+z \\ z \end{bmatrix} \qquad \text{and} \qquad T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x-y+z \\ y-z \\ z \end{bmatrix}.$$

(a) Show that S and T are inverses by showing $(S \circ T)(\mathbf{v}) = \mathbf{v}$ and $(T \circ S)(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^3$.

(b) Let A be the matrix associated to S and let B be the matrix associated to T. Write down A and B and verify that A and B are inverses of each other.

5. Recall that $(AB)^T = B^T A^T$.

(a) Verify this property is true for the matrices $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}$.

(b) Let A be an $m \times n$ matrix and $\mathbf{v} \in \mathbb{R}^n$. Show that $(A\mathbf{v})^T = \mathbf{v}^T A^T$.

(c) Now let's prove the general property. Fill in the blanks.
Let $B = \begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_n \end{bmatrix}$. Then

$$(AB)^T = \begin{bmatrix} \quad \end{bmatrix}^T = \begin{bmatrix} (\quad)^T \\ \vdots \\ (\quad)^T \end{bmatrix} = \begin{bmatrix} \quad \\ \vdots \\ \quad \end{bmatrix} = \begin{bmatrix} \quad \\ \vdots \\ \quad \end{bmatrix} A^T = B^T A^T$$

6. Prove that $(A^T)^{-1} = (A^{-1})^T$.